Problem I

a) We need to find a $G(x,y,\xi,\eta)$ s.t.

- G is continuous for $(x,y) \in \overline{\mathbb{R}^2_+} \setminus \{(\xi,\eta)\}$
- $G(x,0,\xi,\eta) = 0$

1 pt per issue

- $G(x,y,\xi,\eta)$ is harmonic for $(x,y) \in \mathbb{R}^2_+ \setminus \{(\xi,\eta)\}$
- $G(x,y,\xi,\eta)$ satisfies

$$\Delta_{(x,y)} G = \delta((x,y)-(\xi,\eta))$$

4 points total

We know that

$$G_1 = \frac{1}{2\pi} \log |(x,y)-(\xi,\eta)|$$

is harmonic for $(x,y) \in \mathbb{R}^2 \setminus (\xi,\eta)$

and satisfies.

2 points

$$\Delta_{(x,y)} G_1 = \delta((x,y)-(\xi,\eta))$$

Observe

2 points

$$G_2 = \frac{1}{2\pi} \log|(x,y)-(\xi,-\eta)|$$

is harmonic for $(x,y) \in \mathbb{R}^2 \setminus \{(\xi,-\eta)\}$

and satisfies

$$\Delta_{(x,y)} G_2 = \delta((x,y)-(\xi,-\eta))$$

Now

$$\frac{1}{2\pi} \log|(x,y)-(\xi,\eta)|\Big|_{y=0} = \frac{1}{2\pi} \log|(x,y)-(\xi,-\eta)|\Big|_{y=0}$$

Hence we set $G = G_1 - G_2$ so

G is harmonic for $(x,y) \neq (\xi,\eta), (\xi,-\eta)$.

and $G(x,0,\xi,\eta) = 0$.

4 points to addition

$$\Delta_{(x,y)} G = \delta((x,y)-(\xi,\eta)) - \delta((x,y)-(\xi,-\eta))$$

if $y>0$ then

$$\Delta_{(x,y)} G = \delta((x,y)-(\xi,\eta))$$

Our Green's function is $\quad$ 2 points to final form.

$$G(x,y,\xi,\eta) = \frac{1}{2\pi}\log|(x,y)-(\xi,\eta)|$$

$$-\frac{1}{2\pi}\log|(x,y)-(\xi,-\eta)| = \frac{1}{4\pi}\log\left(\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}\right)$$

The solution to the Boundary value problem is $\quad$ 3 points formula

$$u(\xi,\eta) = \int_{-\infty}^{\infty}\int_{0}^{\infty} G(x,y,\xi,\eta)\, f(x,y)\, dy\, dx$$

$$+ \int_{-\infty}^{\infty} \frac{\partial G}{\partial y}(x,0,\xi,\eta)\, g(x)\, dx.$$

$$u(\xi,\eta) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} \log\left(\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}\right) dy\,dx$$

$$-\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\eta}{(x-\xi)^2+\eta^2}\, g(x)\,dx.$$

+1 point.

Problem 1b

$+\Delta u = f$ with $f > 0$

implies $\Delta u \geq 0 \Rightarrow -\Delta u \leq 0$ 3 points

So u is subharmonic on the ball. u obtains its maximum on the bounday of the ball and this maximum is equal to g.

2 points for theorem.

Problem (2)

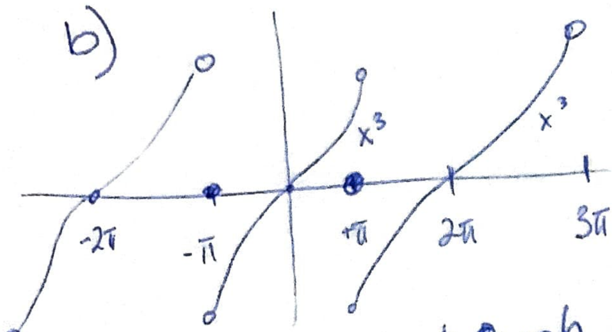
a) Fourier Series of f is given by

integration by parts

3pts $\frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin(nx)\,dx = \frac{6(\pi^2 n^2 - 2)\sin \pi n - 2\pi n(\pi^2 n^2 - 6)\cos(\pi n)}{n^4 \pi}$

3pts $\frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \cos(nx)\,dx = 0$ (odd fuction)

b)



2pts graph

$\frac{f(-\pi) + f(\pi)}{2}$

$= \frac{(-\pi)^3 + \pi^3}{2} = 0.$

2 pts conclusion.

The FS converges to the fuction which is x^3 on every interval of length 2π and 0 on the endpoints.

The series converges pointwise as it is everywhere piecewise continuous when considered as a periodic extension of x^3. 3 pts

1pt

The series <u>doesnt</u> converge uniformly.

$L^2([-\pi,\pi])$ convergence is also true since $x^3 \in L^2([-\pi,\pi])$.

1pt.

Problem 3

a) $\partial_t u = \Delta u + \alpha u$

Let $w = u_1 - u_2$ then if $u_1 \neq u_2 \Rightarrow$

2pts

$$\partial_t w = \Delta w + \alpha w$$

$$\frac{d}{dt} \mathcal{E}(t) = \int_0^1 \Delta w \, \bar{w} \, dt + \int_0^1 w^2 \alpha \, dt$$

where $\mathcal{E}(t) = \int_0^1 w^2 dt$

Using Green's theorem 3pts

$$\frac{d}{dt} \mathcal{E}(t) = -\int_0^1 |\nabla w|^2 dt + \cancel{\int_0^1 w \frac{\partial w}{\partial n} ds} \to 0$$

since $u(0) = u(1) = 0$

$$+ \mathcal{E}(t)\alpha.$$

Then $\frac{d}{dt}\mathcal{E}(t) \leq 0$ if

$$\alpha < 0 \text{ and } 0 \leq \alpha \leq \frac{\int_0^1 |\nabla w|^2 dt}{\int_0^1 |w|^2 dt}$$

1pt 1pt

The constant $\dfrac{\int_0^1 |\nabla w|^2 dt}{\int_0^1 |w|^2 dt}$ is known as the Poincaré Constant.

When $\frac{d}{dt} \mathcal{E}(t) \leq 0 \Rightarrow \mathcal{E}(t)$ is constant

But $\mathcal{E}(0) = 0$. Therefore $u_1 = u_2$.

for this choice of α. 3 pts to conclusion

b) $\partial_t u = \Delta u - u$.

Take the separated solution

$u(t,x) = X(x)T(t)$

then we have $XT' = X''T - XT$ $\boxed{1pt}$

and $\frac{T'}{T} = \frac{X''}{X} - 1$ 2pts

We-rewrite this as

$\boxed{3 pts}$

$\frac{T'}{T} + 1 = \frac{X''}{X} = \lambda$ $\quad X(0) = X(1) = 0$

take eigenvalues $(n\pi)^2 = \lambda$ $\quad n = 1, 2, 3 \ldots$

The eigenfunctions are $X_n(x) = \sin(n\pi x)$

the t part gives the ODE

$$\frac{T'}{T} = -\lambda - 1$$

This has solution $T = Ae^{-(\lambda+1)T}$ $\boxed{2pts}$

The general solution is

$$u(t,x) = \sum_{n=1}^{\infty} A_n e^{-(\lambda+1)t} \sin(n\pi x)$$ $\boxed{2pts}$

if $t=0$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

$A_1 = 3$ $A_3 = 1/7$ all other A_n

are zero. $\lambda_1 = \pi^2$ $\lambda_3 = 9\pi^2$

$$u(t,x) = 3e^{-(\lambda_1+1)t} \sin(\pi x) +$$

$$\frac{1}{7} e^{-(\lambda_3+1)t} \sin(3\pi x)$$

$\boxed{2pts}$ to final form

c) $\partial_t u = \Delta u + \alpha u$

taking the Fourier transform in x gives

$$\left.\begin{aligned}\partial_t \hat{u} &= -\xi^2 \hat{u} + \alpha \hat{u} \\ \text{Then } \hat{u} &= e^{-\xi^2 t + \alpha t}\,\hat{u}_0\end{aligned}\right\} \text{ 3pts}$$

Here $u(0,x) = e^{-|x|}$. The Fourier transform of $u(0,x)$ is

$$\hat{u}_0 = \frac{2}{1+\xi^2} \Rightarrow \left.\hat{u}(t,\xi) = \frac{2e^{-t\xi^2+\alpha t}}{1+\xi^2}\right\} \text{ 2pts}$$

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2e^{-t\xi^2+\alpha t}}{1+\xi^2}\, e^{ix\cdot\xi}\, d\xi \quad \text{2pts}$$

$$\left.\begin{aligned}|u(t,x)| &\leq \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}_0\, e^{-t\xi^2+\alpha t}\, e^{ix\cdot\xi}|\, d\xi \\ &\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}_0|\, d\xi \ \sup_{\xi} |e^{-t\xi^2+\alpha t}|\end{aligned}\right\} \text{ 2pts}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+\xi^2}\, d\xi = 1$$

$$\sup_{\xi} |e^{-t\xi^2+\alpha t}| \to 0 \quad \forall \xi \text{ as } t \to \infty.$$

Thus $|u(t,x)| \to 0$ as $t \to \infty$.

1 pts to conclusion

Problem 3/

$$2u_{xx} - 4u_{xy} - u_{yy} = 0$$

$$|A| = \left|\begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix}\right| = -2-(-2)^2 > 0$$

3pts

This is hyperbolic

$$(\sqrt{2}\partial_x - \sqrt{2}\partial_y)^2 = 2\partial_x - 2(\sqrt{2})\sqrt{2})\partial_y\partial_x + 2\partial_y$$
$$= 2\partial_x - 4\partial_y\partial_x + 2\partial_y \quad \Big\} \text{ 3pts}$$

$$(\sqrt{2}\partial_x - \sqrt{2}\partial_y)^2 - 3\partial_y^2 = 0 \quad \Big\} \text{ 2pts}$$

Change of variables

$$\sqrt{2}(x+y) = \xi$$

$$\sqrt{3}Y = \eta$$

2pts

gives canonical form

$$(\partial_\xi^2 - \partial_\eta^2)u(\xi,\eta) = 0$$

$u(0,\xi) = e^{-\xi^2}$ $\quad \partial_\eta u(0,\xi) = 0$

D'Alembert's principle $\left(\begin{matrix} g = 0 \\ f = e^{-\xi^2} \end{matrix}\right)$ 2pts

$\eta = t \quad x = \xi$

gives

$$u(\eta,\xi) = \frac{e^{-(\xi-\eta)^2} + e^{-(\xi+\eta)^2}}{2} \Bigg\} \text{ 5 points}$$

Then

$$\frac{e^{-(\sqrt{2}(x+y)-\sqrt{3}y)^2} + e^{-(\sqrt{2}(x+y)+\sqrt{3}y)^2}}{2} = u(t,x).$$

3 points
to conclusion